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On the two q-analogue logarithmic functions: $\ln_q(w)$, $\ln\{e_q(z)\}$

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Abstract. There is a simple, multi-sheet Riemann surface associated with $e_q(z)$'s inverse function $\ln_q(w)$ for $0 < q \leqslant 1$. A principal sheet for $\ln_q(w)$ can be defined. However, the topology of the Riemann surface for $\ln_q(w)$ changes each time q increases above the collision point q^*_τ of a pair of the turning points τ_i of $e_q(x)$. There is also a power series representation for $\ln_q(1+w)$. An infinite-product representation for $e_q(z)$ is used to obtain the ordinary natural logarithm $\ln\{e_q(z)\}$ and the values of the sum rules $\sigma^e_n \equiv \sum_{i=1}^{\infty}(1/z_i)^n$ for the zeros z_i of $e_q(z)$. For $|z| < |z_1|$, $e_q(z) = \exp\{b(z)\}$ where $b(z) = -\sum_{n=1}^{\infty}\frac{1}{n}\sigma^e_n z^n$. The values of the sum rules for the q-trigonometric functions, σ^c_{2n} and σ^s_{2n+1}, are q-deformations of the usual Bernoulli numbers.

1. Introduction

The ordinary exponential and logarithmic functions find frequent and varied applications in all fields of physics. Recently in the study of quantum algebras, the q-exponential function [1] or mapping $w = e_q(z)$ has reappeared [2–4] from a rather dormant status in mathematical physics. This order-zero entire function can be defined by

$$e_q(z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{[n]!} \tag{1}$$

where

$$[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}. \tag{2}$$

The series in (1) converges uniformly and absolutely for all finite z. Since $[n]$ is invariant under $q \to 1/q$, for real q it suffices to study $0 < q \leqslant 1$. The q-factorial is defined by $[n]! \equiv [n][n-1]\ldots[1]$, $[0]! \equiv 1$. As $q \to 1$, $e_q(z) \to \exp(z)$, the ordinary exponential function.

In [5], we reported some of the remarkable analytic and numerical properties of the infinity of zeros, z_i, of $e_q(x)$ for $x < 0$. In particular, as q increases above the first collision point at $q^*_z \approx 0.14$, these zeros collide in pairs and then move off into the complex z plane; see figure 1. They move off as (and remain) a complex conjugate pair $\mu_{A,\bar{A}}$. The turning points of $e_q(z)$, i.e. the zeros of the first derivative $e'_q(z) \equiv \mathrm{d}e_q(z)/\mathrm{d}x$, behave in a similar manner. For instance, at $q^*_\tau \approx 0.25$ the first two turning points, τ_1 and τ_2, collide and move off as a complex conjugate pair $\tau_{A,\bar{A}}$.

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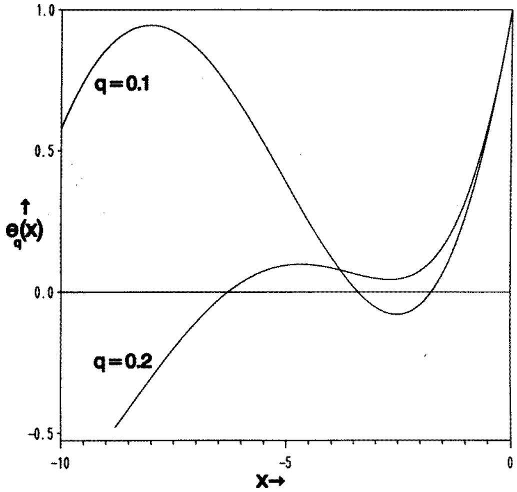

Figure 1. Plot showing the behaviour of the q-analogue exponential function $e_q(x)$ for x negative. The $q = 0.1$ curve displays the universal behaviour of $e_q(x)$ for $q < q_1^*$ ($q_1^* \approx 0.14$). As q increases above the first collision point at $q_1^* \approx 0.14$, the zeros, $\mu_i = z_i$, collide in pairs and then move off into the complex z plane. They move off as (and remain) a complex conjugate pair. The $q = 0.2$ curve displays the behaviour of $e_q(x)$ after the collision of the first pair of zeros μ_1, μ_2 but before the collsion of the first pair of turning points. The first two turning points τ_1, τ_2 collide at $q_{\tau 1}^* \approx 0.25$. The turning points τ_i of $e_q(z)$ are mapped into the branch points b_i, of $\ln_q(w)$.

In this paper, we first show that there is a simple, multi-sheet Riemann surface associated with $w = e_q(z)$'s inverse function $z = \ln_q(w)$. As with the usual $\ln(w)$ function, the Riemann surface of $z = \ln_q(w)$ defines a single-valued map onto the entire complex z plane. Also, as in the usual case when $q = 1$, a principal sheet for $z = \ln_q(w)$ can be defined. However, unlike for the ordinary $\ln(w)$ and $\exp(z)$, the topology of the Riemann surface for $\ln_q(w)$ changes each time q increases above the collision point q_τ^* of a pair of the turning points τ_i of $e_q(z)$. The turning points of $e_q(z)$ can be used to define square-root branch points of $\ln_q(w)$ in the complex w plane, i.e. $b_i = e_q(\tau_i)$.

In section 3, we obtain a power series representation for $\ln_q(1 + w)$.

In the mathematics and physics literature†, one also finds the exponential function $E_q(z)$ defined by Jackson [7, 8]. It is also given by (1) but with $[n]$ replaced by $[n]_J$ where

$$[n]_J = q^{(n-1)/2}[n] = \frac{1 - q^n}{1 - q}. \tag{3}$$

For $q > 1$, $E_q(z)$ has simpler properties‡ than $e_q(z)$. We also construct the Riemann surface for its inverse function $\mathrm{Ln}_q(w)$. With the substitution $[n] \to [n]_J$, the power series representation for $\ln_q(1 + w)$ also holds for $\mathrm{Ln}_q(1 + w)$.

In section 4, we use the infinite-product representation [5] for $e_q(z)$ (i) to obtain the ordinary natural logarithm $\ln\{e_q(z)\}$, and (ii) to evaluate for arbitrary integer $n > 0$ the sum

† Recent reviews of quantum algebras are listed in [6].

‡ For $0 < q < 1$, $E_q(z)$ is a meromorphic function whose power series converges uniformly and absolutely for $|z| < (1-q)^{-1}$ but diverges otherwise. However, by the relation, $E_s(x)E_{1/s}(-x) = 1$ for s real, results for $q > 1$ can be used for $0 < q < 1$, see [5].

rules

$$\sigma_n^e \equiv \sum_{i=1}^{\infty} \left(\frac{1}{z_i}\right)^n \tag{4}$$

for the zeros z_i of $e_q(z)$. Therefore, for c-number arguments,

$$e_q(x)e_q(y) = \exp\{b(x) + b(y)\} \tag{5}$$

where $b(x)$ is defined below in (20). For $|z| < |z_1|$ the modulus of the first zero,

$$b(z) = -\sum_{n=1}^{\infty} \frac{1}{n}\sigma_n^e z^n. \tag{6}$$

We also obtain the logarithms and values of the associated sum rules for all derivatives and integrals of $e_q(x)$, and for the associated q-trigonometric functions [1, 5] $\cos_q(z)$ and $\sin_q(z)$. These results also hold for the analogous functions involving $[n]_J$.

Section 5 contains some concluding remarks. In particular, the values of the sum rules for the q-trigonometric functions, σ_{2n}^c and σ_{2n+1}^s, are q-deformations of the usual Bernoulli numbers.

2. Riemann surfaces of q-analogue logarithmic functions $\mathrm{ln}_{q(w)}$ and $\mathrm{Ln}_{q(w)}$

For two reasons, we begin by first analysing the Riemann surface associated with the mapping of Jackson's exponential function $w = u + \mathrm{i}v = E_q(z)$ and of its inverse $z = x + \mathrm{i}y = \mathrm{Ln}_q(w)$. First, the generic structure of the Riemann surface for $\mathrm{Ln}_q(w)$ for $q^E > 1$ is the same as that for $\ln_q(w)$ for $q^e < q_1^*(q_1^* \approx 0.14)$. Second, as q^e varies the topology of the Riemann surface changes for $\ln_q(w)$ but the topology remains invariant for $\mathrm{Ln}_q(w)$ for all $q^E > 1$. Normally we will suppress the superscripts E or e on the q's.

2.1. Riemann surface for $\mathrm{Ln}_q(w)$

Figures 2 and 3 show the Riemann sheet structure and the mappings of Jackson's exponential function $w = E_q(z)$ and of its inverse $z = \mathrm{Ln}_q(w)$ for $q^E \approx 1.09$. These figures suffice for illustrating the Riemann sheet for all $q > 1$ because the zeros and turning points of $E_q(z)$ do not collide, but simply move along the negative x-axis and out to infinity as $q \to 1$.

These figures also illustrate the Riemann surface for $w = e_q(z)$ and $z = \ln_q(w)$ but only prior to the collision of the first pair of zeros at $q \approx 0.14$.

Notice that the imaginary part $\mathrm{Im}\{e_q(z)\} = 0$ on all full contour lines in figure 2(*b*) whereas the real part $\mathrm{Re}\{e_q(z)\} = 0$ on all broken contour lines. The turning points in the complex z plane are denoted by small full squares, whereas their associated branch points in w are denoted by small full circles (figure 3).

Numerically, for $q^E \approx 1.09$, the first four zeros of $E_q(z)$ are located at -12.1111, -13.2011, -14.3892, -15.6842. The first four turning points and $\mathrm{Ln}_q(w)$'s branch points (b_i in 10^{-11} units) are respectively at $(\tau_i, b_i) = (-12.4, -43)$, $(-13.6, 5.0)$, $(-14.9, -1.8)$, $(-16.3, 4.4)$. Since $q^E \approx 1$, the asymptotic formula in [5] for τ_i^E is a bad approximation for these values.

Figures for the lower sheets of a Riemann surface w are omitted in this paper since they simply have the conjugate structures, per the Schwarz reflection principle.

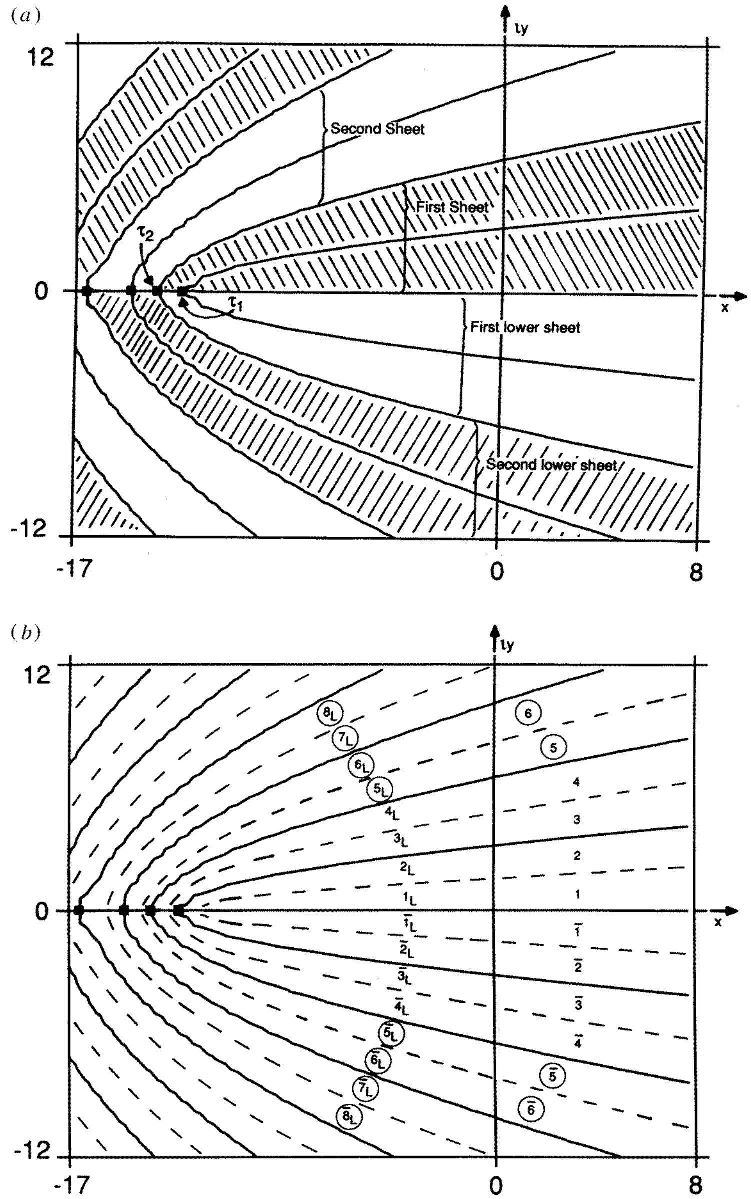

Figure 2. These two figures and figures 3(*a*) and (*b*) show the Riemann sheet structure and the mappings of Jackson's exponential function $E_q(z)$ and of its inverse function $\mathrm{Ln}_q(w)$ for $q^E = 1.09$. For instance, $w = E_q(z)$ maps the region labelled '$1, 2, 1_L, 2_L$' in figure 2(*b*) onto the upper-half-plane (uhp) of the first w sheet for $\mathrm{Ln}_q(w)$; see figure 3(*a*). The turning points τ_1, τ_2 are mapped respectively into the branch points b_1, b_2 of figure 3(*a*). These figures suffice to illustrate the behaviour of $E_q(z)$ and $\mathrm{Ln}_q(w)$ for all $q^E > 1$ because as $q^E \to 1$, the zeros and turning points of $E_q(z)$ do not collide, but simply move along the negative x-axis and out to infinity. In the complex w plane the associated branch points of $\mathrm{Ln}_q(w)$ all move into the origin. This limit thereby gives the usual Riemann surface for $\exp(z)$ and $\ln(w)$. Figures 2 and 3 also illustrate the Riemann surface for $e_q(z)$ and $\ln_q(w)$ but only prior to the collision of the first pair of zeros, i.e. for $q < q_1^*$ ($q_1^* \approx 0.14$). Figures 4–8 show the Riemann surfaces of $e_q(z)$ and $\ln_q(w)$ for larger q values, $q_1^* < q \leqslant 1$.

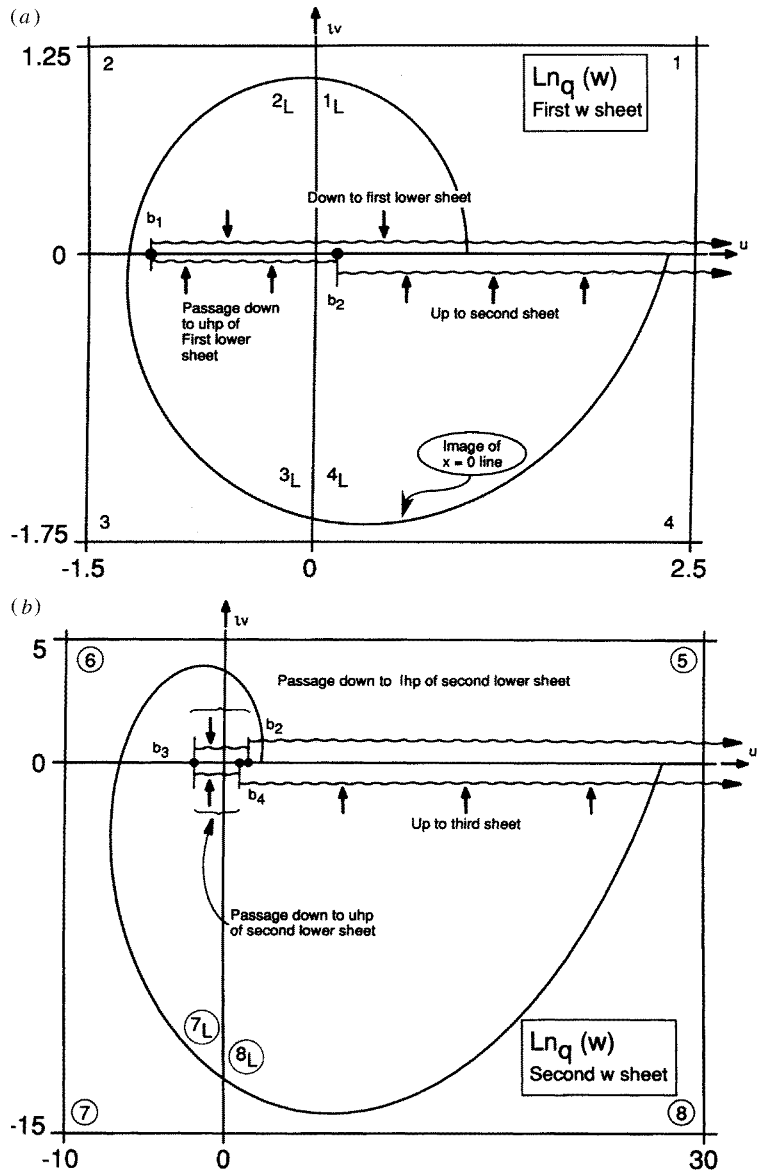

Figure 3. (*a*) The first upper sheet of $\mathrm{Ln}_q(w)$ for $q^E = 1.09$. The turning points τ_1, τ_2 in figure 2 for $E_q(z)$ are mapped respectively into the square-root branch points b_1, b_2 of figure 3(*a*) and (*b*) for $\mathrm{Ln}_q(w)$. An 'opening spiral', instead of the usual unit circle, is the 'image' of the positive y-axis (the $x = 0$ line) in figure 2. The first lower sheet of $\mathrm{Ln}_q(w)$ is the mirror image of this figure (the reflection is through the horizontal u-axis); the lower sheets corresponding to the other 'upper sheet' figures in this paper are similarly obtained. (*b*) The second upper sheet of $\mathrm{Ln}_q(w)$ for $q^E = 1.09$. Note that the opening spiral continues that in (*a*). The cut above the real axis from b_2 to ∞ goes back down to the first sheet, figure 3(*a*).

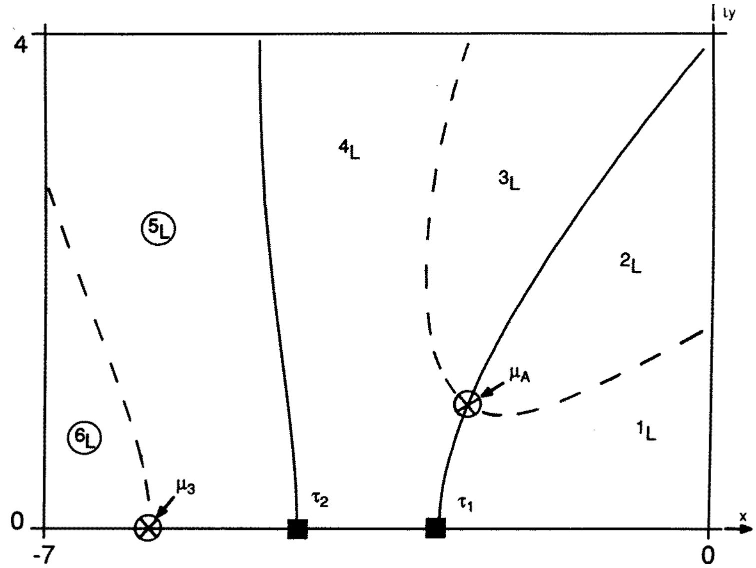

Figure 4. This figure and figure (5) show the Riemann sheet structure and the mappings of $e_q(z)$ and of its inverse function $\ln_q(w)$ for $0.14 < q \approx 0.22 < 0.25$. For this range of q, the first two zeros μ_1, μ_2 of $e_q(x)$ have collided and have moved off as a complex conjugate pair $\mu_A, \mu_{\bar{A}}$; the μ_A zero is marked in this figure. Note that as in figure 2, $\mathrm{Im}\{e_q(z)\} = 0$ on all full contour lines, whereas $\mathrm{Re}\{e_q(z)\} = 0$ on all broken coutour lines.

2.2. Riemann surface for $\ln_q(w)$*:*

For $q \lesssim 0.14$, figures 1–3 also show the topology and branch point structure for the mappings $w = e_q(z)$ and its inverse $z = \ln_q(w)$.

Figures 4 and 5 are for after the collision of the first pair of zeros of $e_q(z)$ but prior to the collision of the first pair of its turning points, so the structure shown is generic for $0.14 < q < 0.25$. Note that $w_A = e_q(\mu_A) = 0$ occurs as an analytic point for $w = e_q(z)$ which is not possible for the ordinary $\exp(z)$ in the finite z plane.

Numerically, figures 4 and 5 are for $q \approx 0.22$; the first two zeros of $e_q(z)$ are located at $\mu_A = -2.51 + \mathrm{i}0.87$, $\mu_{\bar{A}} = \bar{\mu}_A$. The first two turning points and $\ln_q(w)$'s branch points (b_i in 10^{-3} units) are respectively at $(\tau_i, b_i) = (-2.6, 47.70), (-4.7, 69.36)$.

Figures 6–8 are for after the collision of the first pair of turning points of $e_q(z)$. The topology of the Riemann surface has a new inter-surface structure due to this collision; the figures and their captions explain this new structure. In particular compared with figure 5, following the collision at $q^*_{\tau 1} \approx 0.25$, there no longer exists the $b_1 - b_2$ passage from the lower half of the principal w sheet to the first lower w sheet. Instead, the b_A passages are to the second upper w sheet.

Numerically, figures 6–8 are for $q \approx 0.35$. The first two zeros of $e_q(z)$ are now located at $\mu_A = -2.8222 + \mathrm{i}1.969$, $\mu_{\bar{A}} = \bar{\mu}_A$; the third zero remains on the negative real axis at $\mu_3 = -5.197\,55$. The first four turning points and $\ln_q(w)$'s branch points (b_i in 10^{-3} units) are respectively at $(\tau_i, b_\mathrm{i}) = (-3.5434 \pm \mathrm{i}1.329\,45, 22.2415 \pm \mathrm{i}18.79), (-6.3471, -9.095\,87), (-10.7028, 87.536)$. In figures 7 and 8, for clarity of illustration, the position of b_A has been displaced from its true position.

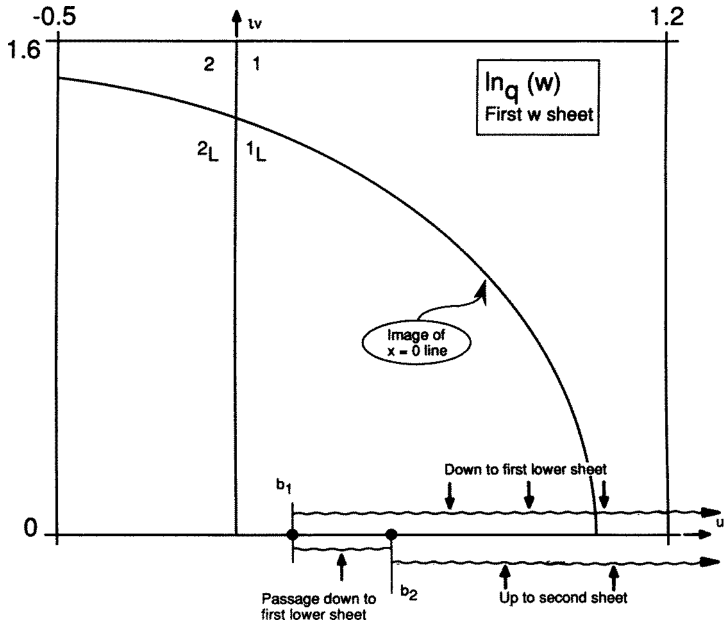

Figure 5. The first upper sheet for $\ln_q(w)$ for $0.14 < q \approx 0.22 < 0.25$. When q is increased to $q \approx 0.25$, the branch points $b_1 = b_2$ coincide since the turning points τ_1, τ_2 of figure 4 have collided. Then, the branch cut to the first lower sheet no longer exists. τ_1, τ_2 become a complex conjugate pair $\tau_A, \tau_{\bar{A}}$ and move off into the complex z plane, as shown in figures 6–8.

3. Power series representations for $\ln_q(1 + w)$ and $\mathrm{Ln}_q(1 + w)$

To obtain the power series for $\ln_q(1 + w)$, we write

$$\begin{aligned} \ln_q(1+w) &= c_1 w + c_2 w^2 + \cdots \\ &= \sum_{n=1}^{\infty} c_n w^n. \end{aligned} \tag{7}$$

Then for $a = \ln_q(1 + w)$,

$$\begin{aligned} e_q^a &= 1 + a + \frac{a^2}{[2]!} + \cdots \\ &= 1 + w. \end{aligned} \tag{8}$$

So by equating coefficients, we find

$$\begin{aligned} c_1 &= 1 \\ c_n &= -\sum_{l=2}^{n} \frac{1}{[l]!} \left\{ \sum_{(k_1,k_2,\ldots,k_l)} c_{k_1} c_{k_2} \ldots c_{k_l} \right\} \qquad n \geqslant 2. \end{aligned} \tag{9}$$

In order to follow later expressions in this paper, it is essential to understand the second summation $\sum_{(k_1,k_2,\ldots,k_l)}$: in it, each k_i = 'positive integer', $i = 1, 2, \ldots, l$. The expression $(k_1, k_2, \ldots, k_l)$ denotes that, for fixed n and l, the summation is the symmetric permutations of each partition of n which satisfy the condition '$k_1 + k_2 + \cdots k_l = n$'.

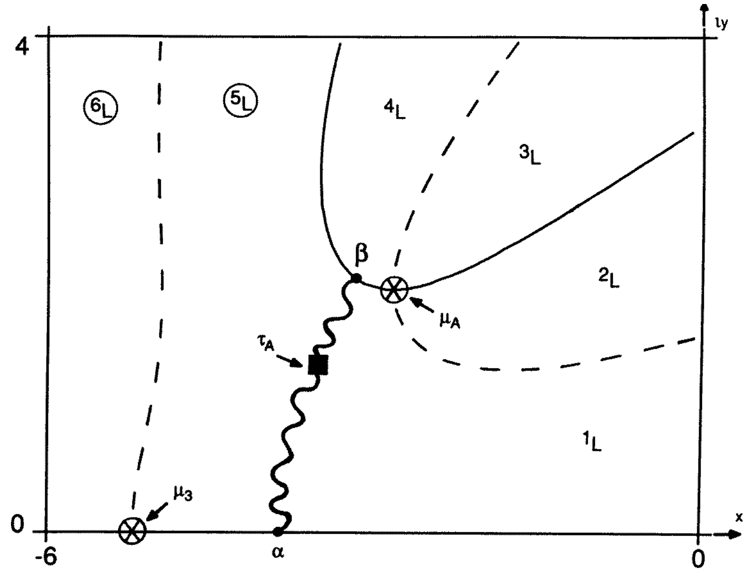

Figure 6. This figure and figures 7 and 8 show the Riemann sheet structure and the mappings of $e_q(z)$ and of its inverse function $\ln_q(w)$ for $q \approx 0.35$. The first two turning points τ_1, τ_2 of $e_q(x)$ have collided and have moved off as a complex conjugate pair τ_A, $\tau_{\bar{A}}$; the τ_A turning point is marked in this figure, $\tau_A = -3.54 + \mathrm{i}1.33$. The line corresponding to the $\alpha'\beta'$ branch cut through b_A, see figures 7 and 8, is the wiggly line from α on the $x < 0$ axis, through τ_A, and on to β on the $\mathrm{Im}\{e_q(z)\} = 0$ curve. τ_A (and b_A) are fixed, but α and β (α' and β') are simple though arbitrary positions on their respective $\mathrm{Im}\{e_q(z)\} = 0$ lines. The third zero μ_3 of $e_q(z)$ is still on the $x < 0$ axis.

For instance, for $n = 4$,

$$\begin{aligned}
&\sum_{(k_1,k_2,k_3,k_4)} c_{k_1}c_{k_2}c_{k_3}c_{k_4} = \{c_1c_1c_1c_1\} = (c_1)^4 \\
&\sum_{(k_1,k_2,k_3)} c_{k_1}c_{k_2}c_{k_3} = \{c_1c_1c_2 + c_1c_2c_1 + c_2c_1c_1\} = 3c_1c_1c_2 \\
&\sum_{(k_1,k_2)} c_{k_1}c_{k_2} = \{c_2c_2\} + \{c_1c_3 + c_3c_1\} = (c_2)^2 + 2c_1c_3.
\end{aligned} \tag{10}$$

This power series for $\ln_q(1 + w)$ is expected to converge only for some w domain, e.g. for $w \leqslant$ ‘modulus of distance to the nearest branch point’. Note that as $q \to 0$, $w = e_q(z) \to w = 1 + z$ and $z = \ln_q(w) \to z = w - 1$, so $e_q\{\ln_q(w)\} \to e_q\{w - 1\} \to w$.

Thus, the first few terms give

$$\begin{aligned}
\ln_q(1 + w) &= w - \frac{1}{[2]!}w^2 - \left\{\frac{1}{[3]!} - \frac{2}{[2]![2]!}\right\}w^3 \\
&\quad - \left\{\frac{1}{[4]!} - \frac{2}{[2]!}\left(\frac{1}{[3]!} - \frac{2}{[2]![2]!}\right) + \left(\frac{1}{[2]!}\right)^3 - \frac{3}{[3]![2]!}\right\}w^4 + \cdots \\
&= w - \frac{1}{[2]!}w^2 - \left\{\frac{1}{[3]!} - 2\left(\frac{1}{[2]!}\right)^2\right\}w^3 \\
&\quad - \left\{\frac{1}{[4]!} - \frac{5}{[3]![2]!} + 5\left(\frac{1}{[2]!}\right)^3\right\}w^4 + \cdots
\end{aligned} \tag{11}$$

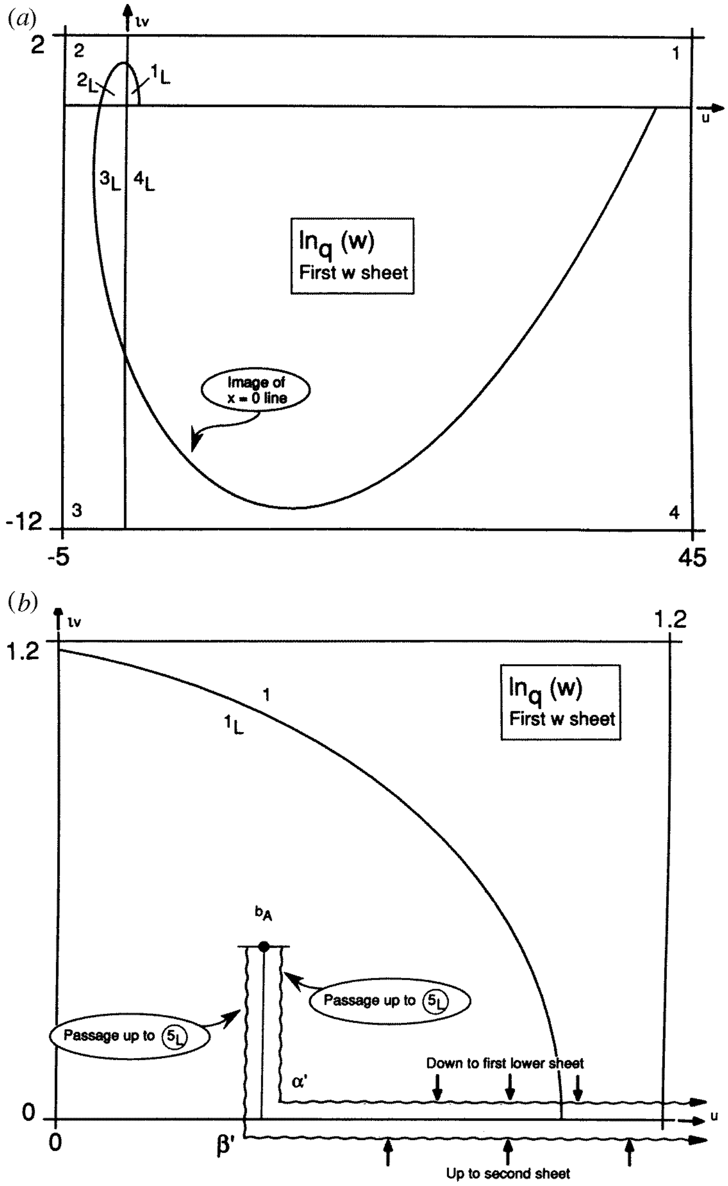

Figure 7. (*a*) The first upper sheet of $\ln_q(w)$ for $q = 0.35$. The image of the $x = 0$ line in the complex z plane is shown. (*b*) An enlargement of the first quadrant which shows the $\alpha'\beta'$ branch cut. For clarity of illustration, the position of b_A has been displaced from its true position at $b_A = 0.0222 + \mathrm{i}0.0188$.

Notice that here the q-derivative operation defines a new function, $d \ln_q(w)/d_q w \equiv \ln_q(w)' \neq 1/w$, because it does *not* yield a known q-special function since

$$\frac{\mathrm{d}}{\mathrm{d}_q w} \ln_q(1+w) = 1 - w - \left\{ \frac{1}{[2]!} - 2[3] \left(\frac{1}{[2]!} \right)^2 \right\} w^2$$

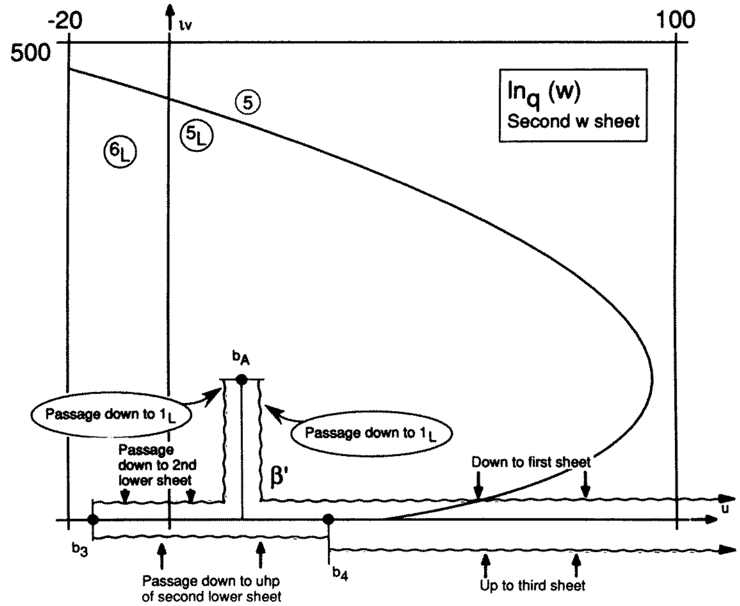

Figure 8. The second upper sheet of $\ln_q(w)$ for $q = 0.35$. The b_A square-root branch point only occurs on the first two upper sheets, i.e. in figure 7 and here. The α' point (not shown) lies opposite the β' point and to the left of the b_A cut structure.

$$-\left\{\frac{1}{[3]!} - \frac{5[4]}{[3]![2]!} + 5[4]\left(\frac{1}{[2]!}\right)^3\right\} w^3 + \cdots \tag{12}$$

unlike [5] for $e_q(z)$, $\cos_q(z)$, and $\sin_q(z)$.

4. Natural logarithms and sum rules for $e_q(z)$ and related functions

By the Hadamard–Weierstrass theorem, it was shown in [5] that the following order-zero entire functions have infinite product representations in terms of their respective zeros:

$$e_q(z) = \prod_{i=1}^{\infty}\left(1 - \frac{z}{z_i}\right) \tag{13}$$

$$e_q^{(r)}(x) \equiv \frac{\mathrm{d}^r}{\mathrm{d}x^r} e_q(x) = \alpha_r \prod_{i=1}^{\infty}\left(1 - \frac{x}{z_i^{(r)}}\right) \qquad r = 1, 2, \ldots$$
$$\alpha_r = \frac{r!}{[r]!} \tag{14}$$

$$\begin{aligned} e_q^{(-r)}(x) &= \int^x \mathrm{d}x_1 \int^{x_1} \mathrm{d}x_2 \ldots \int^{x_r} \mathrm{d}x_r e_q(x_r) + \text{poly deg } (r-1) \qquad r \geqslant 1 \\ &\equiv \sum_{n=0}^{\infty} \frac{n!}{(n+r)!} \frac{x^{n+r}}{[n]!} \\ &= \left(\frac{x^r}{r!}\right) \prod_{i=1}^{\infty}\left(1 - \frac{x}{z_i^{(-r)}}\right) \end{aligned} \tag{15}$$

$$\cos_q(z) \equiv \sum_{n=0}^{\infty}(-)^n \frac{z^{2n}}{[2n]!} = \prod_{i=1}^{\infty}\left(1-\left(\frac{z}{c_i}\right)^2\right) \tag{16}$$

$$\sin_q(z) \equiv \sum_{n=0}^{\infty}(-)^n \frac{z^{2n+1}}{[2n+1]!} = z\prod_{i=1}^{\infty}\left(1-\left(\frac{z}{s_i}\right)^2\right). \tag{17}$$

4.1. Derivation of $\ln\{e_q(z)\}$ *and of the values of* $\sigma_n^e \equiv \sum_{i=1}^{\infty}(1/z_i)^n$

By taking the ordinary natural logarithm of

$$e_q(z) = \prod_{i=1}^{\infty}\left(1-\frac{z}{z_i}\right) \tag{18}$$

we obtain

$$\ln\{e_q(z)\} = \sum_{i=1}^{\infty}\ln\left\{1-\frac{z}{z_i}\right\} = -z\left\{\sum_{i=1}^{\infty}\left(\frac{1}{z_i}\right)\right\} - \frac{z^2}{2}\left\{\sum_{i=1}^{\infty}\left(\frac{1}{z_i}\right)^2\right\} - \frac{z^3}{3}\left\{\sum_{i=1}^{\infty}\left(\frac{1}{z_i}\right)^3\right\}\cdots = b(z) \tag{19}$$

where the function

$$b(z) \equiv \sum_{i=1}^{\infty}\ln\left\{1-\frac{z}{z_i}\right\} = -\sum_{n=1}^{\infty}\frac{1}{n}\sigma_n^e z^n \qquad |z| < |z_1|. \tag{20}$$

Figure 7 of [5] shows the polar part $\rho_i = |z_i|$ of the first eight zeros of $e_q(z)$ for $\approx 0.1 < q <\approx 0.95$. Note that $\rho_i > \rho_{i-1} \geqslant \rho_1$ where ρ_1 is the modulus of the first zero. The function $b(z) = \ln\{e_q(z)\}$ is thereby expressed in terms of the sum rules for the zeros of $e_q(z)$ since

$$\sigma_n^e \equiv \sum_{i=1}^{\infty}\left(\frac{1}{z_i}\right)^n \qquad n = 1, 2, \ldots. \tag{21}$$

By (20), the multi-sheet Riemann surface of $b(z) = \ln\{e_q(z)\}$ consists of logarithmic branch points at the zeros, z_i, of $e_q(z)$.

The basic properties of $e_q(x)$ displayed in figure 1 for $q = 0.1$ follow simply from these expressions for $b(u)$. For instance, the zeros of $e_q(x)$ correspond to where $b(u)$ diverges. A sign change of $e_q(x)$ is due to the principal-value phase change of '$+\mathrm{i}\pi$' at the branch point of $\ln\{1 - z/z_i\}$.

Next, to evaluate these sum rules we proceed as in the above derivation of the power series representation for $\ln_q(1 + w)$. We simply expand both sides of

$$e_q(z) = e^{b(z)}$$
$$1 + \frac{z}{[1]!} + \frac{z^2}{[2]!} + \cdots = 1 + \frac{b}{1!} + \frac{b^2}{2!} + \cdots. \tag{22}$$

Equating coefficients then gives a recursive formula† for these sum rules:

$$\sigma_1^e = -1$$
$$\sigma_n^e = n\left\{\sum_{l=2}^{n} \frac{(-)^l}{l!}\left(\sum_{(k_1,k_2,\cdots k_l)} \frac{\sigma_{k_1}\sigma_{k_2}\cdots\sigma_{k_l}}{k_1k_2\ldots k_l}\right) - \frac{1}{[n]!}\right\} \qquad n \geqslant 2. \tag{23}$$

The notation in the second summation is explained following (9) for $\ln_q(1+w)$.

The first such sum rules are:

$$\begin{aligned} \sigma_1^e &= -1 \\ \sigma_2^e &= 1 - \frac{2}{[2]!} \\ \sigma_3^e &= -1 + \frac{3}{[2]!} - \frac{3}{[3]!} \\ \sigma_4^e &= 1 - \frac{4}{[2]!} + \frac{4}{[3]!} - \frac{4}{[4]!} + \frac{2}{[2]![2]!}. \end{aligned} \tag{24}$$

The values of σ_n^e can also be directly obtained from

$$\sigma_n^e = n\sum_{l=1}^{n} \frac{(-)^l}{l}\left\{\sum_{(k_1,k_2,\cdots k_l)} \frac{1}{[k_1]![k_2]!\ldots[k_l]!}\right\}. \tag{25}$$

Equation (25) follows by expanding (19):

$$b(z) = -\sum_{n=1}^{\infty} \frac{1}{n}\sigma_n^e z^n = \ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} + \cdots \tag{26}$$

where

$$y = e_q(z) - 1 = \frac{z}{[1]!} + \frac{z^2}{[2]!} + \frac{z^3}{[3]!} + \cdots \tag{27}$$

and then equating coefficients of z^n.

Equivalently, these formulae can be interpreted as representations of the reciprocals of the 'bracket' factorials in terms of sums of the reciprocals of the zeros of $e_q(z)$:

$$\begin{aligned} \frac{1}{[2]!} &= \frac{1}{2!} - \frac{1}{2}\sigma_2^e \\ \frac{1}{[3]!} &= \frac{1}{3!} - \frac{1}{2}\sigma_2^e - \frac{1}{3}\sigma_3^e \\ \frac{1}{[4]!} &= \frac{1}{4!} - \frac{1}{4}\sigma_2^e - \frac{1}{3}\sigma_3^e - \frac{1}{4}\sigma_4^e + \frac{1}{8}(\sigma_2^e)^2. \end{aligned} \tag{28}$$

The results in this subsection also give $\ln\{E_q(z)\}$ for the analogous $E_q(z)$ for $q > 1$ by the substitution $[n] \to [n]_J$.

4.2. Logarithms and sum rules for related q-analogue functions:

(i) For the 'rth' derivative of $e_q(x)$, $e_q^{(r)}(x) \equiv \frac{\mathrm{d}^r}{\mathrm{d}x^r} e_q(x)$, we similary obtain ($\alpha_r \equiv \frac{r!}{[r]!}$)

$$\ln\{e_q^{(r)}(x)\} = \ln\alpha_r + b^{(r)}(x) \qquad r = 1, 2, \ldots$$
$$b^{(r)}(z) = \sum_{i=1}^{\infty} \ln\left(1 - \frac{z}{z_i^{(r)}}\right) \tag{29}$$

† These σ_n^e sum rules can also be evaluated [5] by expanding both sides of an infinite-product representation of $e_q(z)$. In this way, from σ_n^e for the first few n, we first discovered the general formulae (23) and (25). Equation (23) describes a pattern similar to that occurring in the reversion (inversion) of power series.

where the sum rules for the zeros of the 'rth' derivative of $e_q(x)$ are

$$\sigma_n^{(r)} \equiv \sum_{i=1}^{\infty}\left(\frac{1}{z_i^{(r)}}\right)^n. \tag{30}$$

The values of these $e_q(z)$ derivative sum rules are

$$\begin{aligned}\sigma_1^{(r)} &= -\frac{r+1}{[r+1]}\\ \sigma_n^{(r)} &= n\left\{\sum_{l=2}^{n}\frac{(-)^l}{l!}\left(\sum_{(k_1,k_2,\ldots,k_l)}\frac{\sigma_{k_1}^{(r)}\sigma_{k_2}^{(r)}\cdots\sigma_{k_l}^{(r)}}{k_1k_2\ldots k_l}\right)-L_n^{(r)}\right\}\end{aligned} \tag{31}$$

where the $L_n^{(r)}$ term is given by

$$L_n^{(r)} = \frac{(n+r)(n+r-1)\ldots(n+1)}{[n+r]!}\frac{1}{\alpha_r} = \frac{(r+n)(r+n-1)\ldots(r+1)}{[r+n][r+n-1]\ldots[r+1]}\frac{1}{n!}. \tag{32}$$

Equivalently,

$$\sigma_n^{(r)} = n\sum_{l=1}^{n}\frac{(-)^l}{l}\left\{\sum_{(k_1,k_2,\ldots,k_l)}L_{k_1}^{(r)}L_{k_2}^{(r)}\ldots L_{k_l}^{(r)}\right\}. \tag{33}$$

Thus, the 'rth' derivative of $e_q(z)$ is

$$e_q^{(r)}(z) = \frac{r!}{[r]!}\exp\{b^{(r)}(z)\} \tag{34}$$

where $b^{(r)}(z) = -\sum_{n=1}^{\infty}\frac{1}{n}\sigma_n^{(r)}z^n$, $|z| < |z_1^{(r)}|$.

(ii) For the 'rth' integral of $e_q(z)$ which is defined in (15), we obtain ($\beta_r \equiv 1/r!$)

$$\begin{aligned}\ln\left\{\frac{e_q^{(-r)}(x)}{x^r}\right\} &= \ln\beta_r + b^{(-r)}(x) \qquad r = 1, 2, \ldots\\ b^{(-r)}(z) &= \sum_{i=1}^{\infty}\ln\left(1-\frac{z}{z_i^{(-r)}}\right)\end{aligned} \tag{35}$$

where the associated sum rules are

$$\sigma_n^{(-r)} \equiv \sum_{i=1}^{\infty}\left(\frac{1}{z_i^{(-r)}}\right)^n. \tag{36}$$

The values of these $e_q(z)$ integral sum rules are

$$\begin{aligned}\sigma_1^{(-r)} &= -\frac{1}{r+1}\\ \sigma_n^{(-r)} &= n\left\{\sum_{l=2}^{n}\frac{(-)^l}{l!}\left(\sum_{(k_1,k_2,\ldots,k_l)}\frac{\sigma_{k_1}^{(-r)}\sigma_{k_2}^{(-r)}\ldots\sigma_{k_l}^{(-r)}}{k_1k_2\ldots k_l}\right)-\frac{r!n!}{(r+n)![n]!}\right\}.\end{aligned} \tag{37}$$

Equivalently,

$$\sigma_n^{(-r)} = n\sum_{l=1}^{n}\frac{(-)^l}{l}\left\{\sum_{(k_1,k_2,\ldots,k_l)}L_{k_1}^{(-r)}L_{k_2}^{(-r)}\ldots L_{k_l}^{(-r)}\right\} \tag{38}$$

where the $L_m^{(-r)}$ expression

$$L_m^{(-r)} \equiv \frac{r!m!}{(r+m)![m]!} \tag{39}$$

is also the $l = 1$ term in (37).

Thus, the 'rth' integral of $e_q(z)$ is

$$e_q^{(-r)}(z) = \frac{z^r}{r!}\exp\{b^{(-r)}(z)\} \tag{40}$$

where $b^{(-r)}(z) = -\sum_{n=1}^{\infty}\frac{1}{n}\sigma_n^{(-r)}z^n$, $|z| < |z_1^{(-r)}|$.

(iii) For the q-trigonometric functions, we obtain for the $\cos_q(z)$ function the representation

$$\begin{aligned} \cos_q(z) &= \exp\{b^c(z)\} \\ b^c(z) &= \sum_{i=1}^{\infty}\ln\left(1-\left(\frac{z}{c_i}\right)^2\right) = -\sum_{n=1}^{\infty}\frac{1}{n}\sigma_{2n}^c z^{2n} \qquad |z| < |c_1| \end{aligned} \tag{41}$$

where

$$\sigma_{2n}^c \equiv \sum_{i=1}^{\infty}\left(\frac{1}{c_i^2}\right)^n. \tag{42}$$

The values of the cosine sum rules are

$$\begin{aligned} \sigma_2^c &= \sum_{i=1}^{\infty}\left(\frac{1}{c_i}\right)^2 = \frac{1}{[2]!} \\ \sigma_4^c &= \sum_{i=1}^{\infty}\left(\frac{1}{c_i}\right)^4 = \left(\frac{1}{[2]!}\right)^2 - \frac{2}{[4]!} \\ \sigma_6^c &= \sum_{i=1}^{\infty}\left(\frac{1}{c_i}\right) = \left(\frac{1}{[2]!}\right)^3 - \frac{3}{[2]![4]!} + \frac{3}{[6]!} \\ \sigma_{2n}^c &= n\left\{\sum_{l=2}^{n}\frac{(-)^l}{l!}\left(\sum_{(k_1,k_2,\ldots,k_l)}\frac{\sigma_{2k_1}^c\sigma_{2k_2}^c\ldots\sigma_{2k_l}^c}{k_1k_2\ldots k_l}\right) - \frac{(-)^n}{[2n]!}\right\}. \end{aligned} \tag{43}$$

Equivalently,

$$\sigma_{2n}^c = n\sum_{l=1}^{n}\frac{(-)^l}{l}\left\{\sum_{(k_1,k_2,\ldots,k_l)}L_{2k_1}^c L_{2k_2}^c\ldots L_{2k_l}^c\right\} \tag{44}$$

where as in the last expression of (43)

$$L_{2m}^c \equiv \frac{(-)^m}{[2m]!}. \tag{45}$$

For the $\sin_q(z)$ function, we find

$$\begin{aligned} \sin_q(z) &= z\exp\{b^s(z)\} \\ b^s(z) &= \sum_{i=1}^{\infty}\ln\left(1-\left(\frac{z}{s_i}\right)^2\right) = -\sum_{n=1}^{\infty}\frac{1}{n}\sigma_{2n+1}^s z^{2n} \qquad |z| < |s_1| \end{aligned} \tag{46}$$

where

$$\sigma_{2n+1}^s \equiv \sum_{i=1}^{\infty}\left(\frac{1}{s_i^2}\right)^n. \tag{47}$$

The values of these sine sum rules are

$$\begin{aligned}
\sigma_3^s &= \sum_{i=1}^{\infty}\left(\frac{1}{s_i}\right)^2 = \frac{1}{[3]!}\\
\sigma_5^s &= \sum_{i=1}^{\infty}\left(\frac{1}{s_i}\right)^4 = \left(\frac{1}{[3]!}\right)^2 - \frac{2}{[5]!}\\
\sigma_7^s &= \sum_{i=1}^{\infty}\left(\frac{1}{s_i}\right)^6 = \left(\frac{1}{[3]!}\right)^3 - \frac{3}{[3]![5]!} + \frac{3}{[7]!}\\
\sigma_{2n+1}^s &= n\left\{\sum_{l=2}^{n}\frac{(-)^l}{l!}\left(\sum_{(k_1,k_2,\ldots,k_l)}\frac{\sigma_{2k_1+1}^s\sigma_{2k_2+1}^s\ldots\sigma_{2k_l+1}^s}{k_1k_2\ldots k_l}\right) - \frac{(-)^n}{[2n+1]!}\right\}.
\end{aligned}\tag{48}$$

Equivalently,

$$\sigma_{2n+1}^s = n\sum_{l=1}^{n}\frac{(-)^l}{l}\left\{\sum_{(k_1,k_2,\cdots k_l)} L_{2k_1+1}^s L_{2k_2+1}^s \cdots L_{2k_l+1}^s\right\}\tag{49}$$

where as in the last expression of (48)

$$L_{2m+1}^s \equiv \frac{(-)^m}{[2m+1]!}\tag{50}$$

5. Concluding remarks

(1) The above sum rules and logarithmic results are representation independent; i.e. they also hold for Jackson's q-exponential function $E_q(z)$, its derivatives, integrals, and also for its associated trigonometic functions $\cos_q(z)$ and $\sin_q(z)$. The only change is that the bracket, or deformed integer, $[n]$ is to be replaced by $[n]_J \equiv \frac{1-q^n}{1-q}$.

Since [7, 5] the zeros of $E_q(z)$ for $q > 1$ are at

$$z_i^E = \frac{q^i}{1-q}\tag{51}$$

simple expressions follow: the values of the associated sum rules are

$$\sigma_n^E \equiv \sum_{i=1}^{\infty}\left(\frac{1}{z_i^E}\right)^n = -\frac{(1-q)^n}{1-q^n} = -\frac{(1-q)^{n-1}}{[n]_J}.\tag{52}$$

A power series representation for the associated natural logarithm is

$$b^E(z) \equiv \ln\{E_q(z)\} = \sum_{i=1}^{\infty}\frac{(1-q)^n}{n(1-q^n)}z^n = \sum_{i=1}^{\infty}\frac{(1-q)^{n-1}}{n[n]_J}z^n \qquad |z| < \left|\frac{q}{1-q}\right|.\tag{53}$$

For both representations, $[n]$ and $[n]_J$, of the derivatives and integrals of $e_q(z)$, and of the $\cos_q(z)$ and $\sin_q(z)$ functions, asymptotic formulae for their associated zeros are given in [5] so simple expressions also follow for their σ_n's and $b(z)$'s in the regions where these asymptotic formulae apply.

(2) Useful checks on the above results and for use in applications of them include:

(i) in the bosonic CS (coherent state) limit $q \to 1$, the normal numerical values must be obtained,

(ii) in the $q \to 0$ limit, results corresponding [9] to fermionic CS's should be obtained (this is a quick, though quite trivial, check),

(iii) by the use of $[n] \to [n]_J \equiv \frac{1-q^n}{1-q}$, the known exact zeros of $E_q(z)$ for $q > 1$ can be used for non-trivial checks. These zeros are at $z_i^E = q^i/(1-q)$.

(3) The determination of the series expansion and a general representation for the usual natural logarithm for the q-exponential function, $b(z) = \ln\{e_q(z)\}$, means that the q-analogue coherent states can now be written in the form of an exponential operator acting on the vacuum state:

$$|z\rangle_q = N(|z|)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{[n]!}}|n\rangle_q = N(|z|)\exp\{b(za^+)\}|0\rangle_q \tag{54}$$

where

$$b(za^+) = \sum_{i=1}^{\infty}\ln\left\{1-\frac{za^+}{z_i}\right\}$$

$$b(za^+) = za^+ - \frac{1}{[2]!}(za^+)^2 - \left\{\frac{1}{[3]!} - 2\left(\frac{1}{[2]!}\right)^2\right\}(za^+)^3 \tag{55}$$

$$-\left\{\frac{1}{[4]!} - \frac{5}{[3]![2]!} + 5\left(\frac{1}{[2]!}\right)^3\right\}(za^+)^4 + \ldots .$$

(4) The successful evaluations and applications of the sum rules for the q-trigonometric functions motivate the following definitions of q-analogue generalizations of the usual Bernoulli numbers:

$$\frac{2^{2n-1}}{(2n)!}B_n^q \equiv \sum_{i=1}^{\infty}\left(\frac{1}{s_i}\right)^{2n}\text{(first kind)} = \sigma_{2n+1}^s \tag{56}$$

$$\frac{2^{2n-1}}{(2n)!}\tilde{B}_n^q \equiv \frac{1}{(2^{2n}-1)}\sum_{i=1}^{\infty}\left(\frac{1}{c_i}\right)^{2n}\text{(second kind)} = \frac{1}{(2^{2n}-1)}\sigma_{2n}^c. \tag{57}$$

Hence, under q-deformation, the usual Bernoulli numbers become the values of the sum rules for the reciprocals of the zeros of the q-analogue trigonometric functions, $\cos_q(z)$ and $\sin_q(z)$. For the Riemann zeta function, these results do not yield a unique definition. However, analogous simple definitions for p complex are

$$\frac{1}{\pi^p}\zeta_q(p) \equiv \sum_{i=1}^{\infty}\left(\frac{1}{s_i}\right)^p \text{ (first kind)} \tag{58}$$

$$\frac{1}{\pi^p}\tilde{\zeta}_q(p) \equiv \frac{1}{(2^p-1)}\sum_{i=1}^{\infty}\left(\frac{1}{c_i}\right)^p\text{(second kind).} \tag{59}$$

Note added in proof. The ordinary natural logarithm of $E_q(z)$ for $0 < q < 1$ is shown to be related to a q-analogue dilogarithm, $\mathrm{Li}_2(z; q)$, in [10] and in the recent survey of q-special functions by Koornwinder [11]: From equation (53) and $E_s(x)E_{1/s}(-x) = 1$, for $0 < q < 1$

$$\ln\left\{E_q\left(\frac{z}{1-q}\right)\right\} = \sum_{i=1}^{\infty}\frac{1}{n(1-q^n)}z^n \equiv \mathrm{Li}_2(z;q) \tag{60}$$

which is identical with (53). Formally [10],

$$\lim_{q\uparrow 1}(1-q)\mathrm{Li}_2(z;q) = \sum_{n=1}^{\infty}\frac{z^n}{n^2} = \mathrm{Li}_2(z) \tag{61}$$

the ordinary Euler dilogarithm. Other recent works on q-exponential functions are in [12].

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